ON LINKAGES FOR TRACING CONIC SECTIONS.

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As the linkages described in this paper are believed to be new methods of tracing conic sections, it is thought worth while to offer a brief account of them.

1. *To trace the orthogonal projection of a given curve; and, in particular, of the circle.*

If the intersection of two planes be made the axis of abscissas, the orthogonal projection upon the second plane of any curve lying in the first is given by multiplying the ordinates of this curve by the cosine of the angle between the planes. Since the smaller of the adjacent dihedral angles is usually taken as the angle of the planes, the angle may vary between the limits zero and $\frac{1}{2}\pi$, and its cosine from unity to zero. Therefore a linkage, to project a curve orthogonally, must reduce the ordinates in any given ratio.

In the linkage of Fig. (1), $AC = BC$; P, Q, and D are points taken on BC, AC, and AC extended, such that $CP = CQ = CD$; thus, QP is parallel to AB; and CR, the medial line of triangle QCP, is perpendicular to QP.

Since, in triangles QCR and QDP, $\dfrac{QC}{QD} = \dfrac{1}{2} = \dfrac{QR}{QP}$, angle QPD is a right angle from the similarity of the triangles, and P lies in the ordinate DS. Moreover P divides the ordinate DS in the ratio $\dfrac{SP}{SD} = \dfrac{AQ}{AD} = \dfrac{BC - CP}{BC + CP}$.

Then P is the orthogonal projection of D when the projecting factor is $\cos\theta = \dfrac{BC - CP}{BC + CP}$. Therefore, if A and B be constrained to move in the right line KL—by sliding or otherwise—and D move on any given curve, a tracing point at P will describe the orthogonal projection of that curve, KL being the line of intersection of the planes, and $\cos^{-1}\dfrac{BC - CP}{BC + CP}$ the angle between them.

It is evidently possible, by varying CP between the limits zero and BC, to give θ any desired value from zero to $\frac{1}{2}\pi$. Again, by taking P on CB produced, the curve can be traced on the opposite side of the axis KL, but not with the same generality within the limits of finite links.

When the curve to be projected is a circle, the point D may be guided in the circumference by a third link pivoted to the base, and P will then describe an ellipse of which the semi-major axis equals the radius of the circle and the semi-minor axis equals the radius of the circle multiplied by the cosine of θ. Since the orthogonal projection of similar coaxal curves is another series of similar coaxal curves, it is possible to trace a series of similar coaxal ellipses by varying the length of the third link.

Again, if P be constrained to move in any given curve, D will describe the curve of which this is the orthogonal projection. If P move in a circle, D will describe an ellipse of which the semi-minor axis equals the radius of the circle, and equals the semi-major axis multiplied by cosine θ.

This linkage would be of considerable value as a drawing instrument, since it can describe any ellipse or series of concentric ellipses, and can change in any desired ratio the ordinates of a given curve. The mechanical adjustments would be simple, but a discussion of this side of the subject would perhaps be out of place in this paper.

Two other somewhat similar, and probably new but less simple linkages for orthogonal projection are shown in Fig. (2) and Fig. (3). A and B move in the right line KL, and $QPRD$ in Fig. (2) is a rhombus, while $QPRC$ in Fig. (3) is a parallelogram in which $QP = QD$; in Fig. (3) CD also equals BC.

2. *To trace any Conic Section.*

The linkage of Fig. (1) in combination with a rhombus can be used to describe any conic section regarded as the locus of a point the distance of which from a given straight line is in a constant ratio to its distance from a fixed point.

The linkage $ACDB$ of Fig. (4) is the same as the linkage of Fig. (1), and A and B move freely, as before, upon KL; the point P dividing the ordinates in a constant ratio. Let P be further constrained to lie in one diagonal of a rhombus, the extremities of the other being pivoted at D and F. Then if F be a fixed point, P describes a conic. For P being in the diagonal of the rhombus is equally distant from D and F; and therefore

$$\frac{PS}{PF} = \frac{PS}{PD} = \frac{BP}{2CP} = \text{constant}.$$

The conic described is an ellipse, parabola, or hyperbola according as $BP \gtreqless 2CP$, i. e. $BP \gtreqless \frac{2}{3} BC$.

Since $\dfrac{BP}{2CP} = \dfrac{BC - CP}{2CP}$, and CP can assume any value from zero to BC, the ratio $\dfrac{PS}{PF}$ can vary from infinity to zero. Therefore the linkage can draw

any conic. From the fact that P divides the ordinates of D proportionally, D is seen to describe a conic of the same species as P, and can if desired be made the tracing point.

Another possible form which is better in some mechanical respects, but not perfectly general, is given by transferring the angle of the rhombus from D to P, and making D the describing point lying in the diagonal of the rhombus. Then $DP = DF$, and $\frac{DS}{DF} = \frac{DP + PS}{DP} = \frac{2CP + BP}{2CP}$. If P is taken between B and C this gives an ellipse; if at B, a parabola; if on CB extended, an hyperbola. This ratio however can become as small as one-half only when BP becomes negatively infinite. This difficulty could be overcome by completing the rhombus on AB as diagonal and making use of that one of the new links whose extremity is at B, to carry P.

In the use of a rhombus to constrain one point to be equidistant from two other points, it is necessary that the diagonal link slide freely at the angles Q and R, Fig. (4). A method of obtaining the same result without sliding is by combining Mr. Kempe's angle bisector* with a straight line motion. This would require the replacing of a linkage of five links with one of not less than eleven links, and a description of it would require too much space.

3. *Description of the Hyperbola by aid of the Peaucellier Cell.*

Let the cell be as in Fig. (5) in which $OB = OA$, $BC = AC$, $DC = DE = GC = GE = \frac{1}{2}AC$. Then $CDGE$ is a rhombus and E lies in the line AB,

$$\therefore \quad OE^2 = OA^2 - AE^2 = OA^2 - (AC^2 - CE^2),$$

$$OE^2 - CE^2 = OA^2 - AC^2 = \text{a constant.}$$

By making OE the abscissa and CE the ordinate this becomes the equation of a rectangular hyperbola.

Let ER be rigidly attached to DE, and EQ to GE, making angles DER and GEQ each a right angle, and let the rhombus $ERSQ$ be completed, its sides being each equal to DE. Then rhombus $ERSQ$ is rhombus $CDEG$ turned through a right angle, and QR equals CE. If SR be extended to P making RP equal SR, P will lie in the perpendicular to OS erected at E, and PE will equal RQ,

$$\therefore \quad OE^2 - EP^2 = \text{constant.}$$

If O be fixed and S guided in a straight line passing through O, the locus of P is a rectangular hyperbola.

* "How to draw a straight Line." A. B. Kempe, p. 40. Or the same in Nature, Vol. 16. Or "Proceedings of the Royal Society." 1873.

It is not however necessary that the rhombus $ERSQ$ should be of the same size as $CDEG$, and thus it is possible to give $\frac{PE}{CE}$ any desired value. The equation for the curve traced by P then becomes $x^2 - ly^2 = c$, which is the general equation of the hyperbola referred to its transverse and conjugate diameters as axes of coordinates.